Pergamon

PII: S0021-8928(97)00010-5

J. Appl. Maths Mechs, Vol. 61, No. 1, pp. 75–82, 1997

0021-8928/97 $24.00+0.00

THE INVERSE BOUNDARY-VALUE PROBLEM FOR AN AIRFOIL WITH A SUCTION SLOT†

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(*Received* 30 *January* 1996)

The problem of constructing an airfoil in a flow of an ideal incompressible fluid for a specified velocity distribution on the contour when there is a suction slot in the airfoil is solved. The boundary of the slot is modelled by a segment of an equipotential with the specified velocity distribution in it. A condition for the velocity at the ends of the slot to be finite is introduced which enables one to construct airfoils which have a smooth flow around them. The complex potential of the flow around a circle with a slot is found in analytic form. Conditions are obtained for the closure of the required contour and the method of quasisolutions is used to satisfy these conditions. An example of the construction of the airfoil contour with a lift coefficient greater than two is presented.

The improvement of the aerodynamic properties of wings by means of suction and blowing devices long ago aroused the interest of investigators, and various mathematical models of such devices are known. If a slot, through which suction or blowing of the flow occurs, is sufficiently narrow, it is modelled by a point singularity (a sink or source). The problem of the flow around a Zhukovskii airfoil when there is a source or a sink in the airfoil has been considered in [1]. A complete solution of the inverse problem for an airfoil with sources and sinks, that is, the problem of constructing its contour using the velocity distribution which is specified on it, has been given [2, pp. 128–133]. However, when a slot is modelled by a point singularity in an airfoil, an infinitely high velocity and infinite rarefaction occur, which are physically impossible.

The problem of the flow of ideal incompressible fluid around a circular cylinder with a slot, with an inlet cross-section which is modelled by an equipotential in the form of an arc of a circle, has been investigated in detail in [3] for the case of suction through a wide slot.

However, in the case of a smooth airfoil, with the exception of its trailing edge, when the inlet cross-section of the slot MN is modelled by an equipotential (we will subsequently call it the slot boundary), the velocity at its ends (at the points M and, in the general case, the point N) also becomes infinite. The velocity at the point N is equal to zero if the flow branches in it (Fig. 1a). In order to obtain finite velocities at the points M and N within the framework of the model of an ideal incompressible fluid, it is necessary that the complex potential $w(z)$ should not have singularities at these points and, consequently, that the angles should be preserved in the case of the conformal mapping of the domain G_z (of the exterior of the airfoil) into the domain G_w in the w-plane. It is therefore necessary to seek a solution in the class of airfoils, the contours of which, including the boundary of the slot, have angles at the points M and N equal to $\pi/2$ and $3\pi/2$, respectively (Fig. 1b). In this case, an infinitely thin lip bounds the slot from above. Such airfoils have been considered in the linearized theory in [4].

Taking account of what has been said, the inverse boundary-value problem for an airfoil with suction of the flow of an ideal incompressible fluid under an infinitely thin lip through a slot, the boundary of which (as in [3]) is modelled by an equipotential, is investigated below. In order to obtain a smooth flow around such an airfoil, the condition that the velocities at the ends of the slot should be finite is introduced. This is analogous to the Zhukovskii–Chaplygin postulate that the velocity at the trailing edge of an airfoil is finite. Viscosity can be taken into account using the boundary-layer model, and one of the methods described, for example, in [5]. Here, it is necessary to postulate that the whole of the boundary layer enters the slot.

1. FORMULATION OF THE PROBLEM

In the physical plane z, a plane steady stream of an ideal incompressible fluid flows smoothly around the required airfoil. The contour L_z of this airfoil, which consists of the smooth, impermeable segments

†*Prikl. Mat. Mekh.* Vol. 61, No. 1, pp. 80–87, 1997.

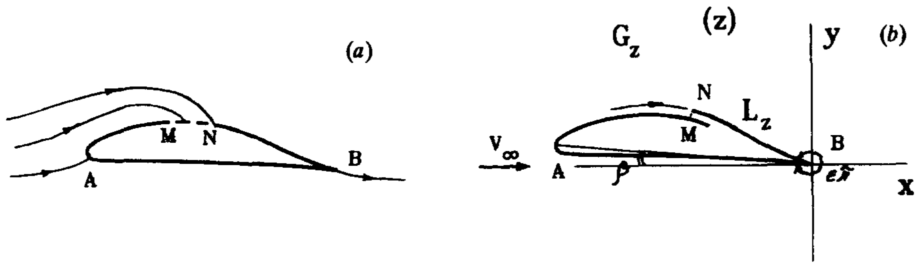

Fig. 1.

BAM and *NB* (*A* is the branch point of the flow and *B* is the point of convergence of the flow) and a permeable segment *MN* (the boundary of the slot), is closed (Fig. 1b). At the trailing edge *B*, which is taken as the origin of the system of coordinates, the angle which is internal to the flow domain is denoted by $\varepsilon\pi$, $1 \leqslant \varepsilon \leqslant 2$. The abscissa x is chosen to be parallel to the specified velocity of the free stream at infinity V_∞ (we shall henceforth assume that the velocities have been made dimensionless by dividing V_∞). The arc abscissa s, made dimensionless by dividing by the perimeter of the contour L_z, is measured from $s = 0$ at the trailing edge up to $s = 1$ such that, as s increases, the flow domain remains to the left.

The tangential flow velocity distribution (Fig. 2a) along the impermeable part of the contour L_z

$$v_\tau = v_\tau(s), \quad s \in [0, s_m) \cup (s_n, 1]$$

is specified, where the piecewise smooth function $v_\tau(s)$ vanishes at the flow branch point A ($s = s_a$) and is continuously differentiable at this point. At the trailing edge (the point B), the velocity $v_\tau(0) = -v_*$, $v_\tau(1) = v_*$ when $\varepsilon = 2$ (an infinitely thin edge) and $v_\tau(0) = v_\tau(1) = 0$ when $1 \leqslant \varepsilon < 2$.

The normal velocity distribution (Fig. 2b)

$$v_n = v_n(s), \quad s \in (s_m, s_n)$$

is specified in the permeable segment *MN* of the contour L_z, where $v_n(s)$ is also a piece-wise smooth function.

As in Golubev's paper [3], we shall assume that the boundary *MN* of the permeable segment of the contour L_z is an equipotential, orthogonal to the direction of the flow velocity (Fig. 1b). Then, $v_\tau(s) = 0$, $s \in (s_m, s_n)$. Since, on approaching the points *M* and *N*, the velocity components which are tangential to the airfoil contour (to the streamlines *AM* and *NB*) must pass continuously into components which are normal to the slot boundary, that is, to the equipotential *MN*, it is necessary, when specifying v (s), to satisfy the conditions

$$|v_\tau(s_m - 0)| = |v_n(s_m + 0)|, \quad |v_\tau(s_n + 0)| = |v_n(s_n - 0)|$$

The airfoil shape and its aerodynamic characteristics now have to be determined.

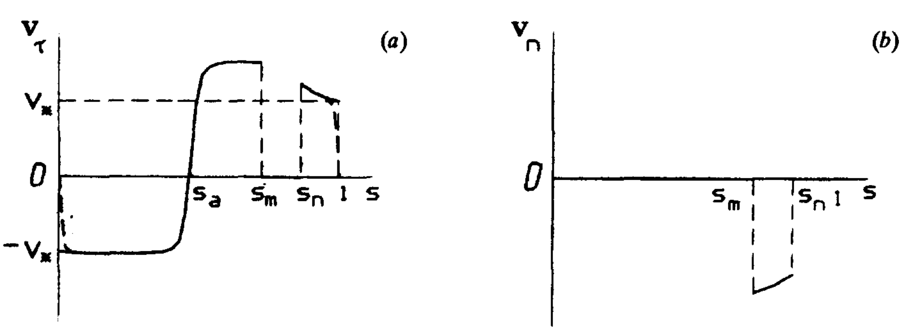

Fig. 2.

Under the assumptions which have been made, a complex flow potential $w(z) = \varphi(x, y) + i\psi(x, y)$ exists, where $z = x + iy$, φ is the velocity potential and Ψ is the stream function.

On the contour L_z, we have

$$\varphi(s) = \int_{s_a}^{s} v_\tau(s)ds, \quad 0 \leqslant s \leqslant 1 \tag{1.1}$$

The velocity circulation Γ is therefore found before solving the problem in the form $\Gamma = \varphi(1) - \varphi(0) = \varphi_1 - \varphi_0$. In the permeable segment MN

$$\psi(s) = -\int_{s_m}^{s} v_n(s)ds, \quad s_m \leqslant s \leqslant s_n \tag{1.2}$$

Consequently, by (1.2), the flow rate Q through the slot MN is determined using the formula $Q = \psi(s_n) - \psi(s_m) = \psi_n$. Since the stream function is constant in the impermeable segment, allowing for the fact that $\psi(s)$ is continuous in L_z, we will have

$$\psi(s) = \begin{cases} 0, & 0 \leqslant s \leqslant s_m \\ -\int_{s_m}^{s} v_n(s)ds, & s_m \leqslant s \leqslant s_n \\ Q, & s_n \leqslant s \leqslant 1 \end{cases} \tag{1.3}$$

Thus, it is required to determine the closed contour L_z and the function $w(z)$ which is analytic in G_z (Fig. 1b) and has the following representation at infinity

$$w(z) = z - \frac{Q - i\Gamma}{2\pi}\ln z + \sum_{k=0}^{\infty} c_k z^{-k}$$

where c_k are complex coefficients, and which satisfies the complex boundary condition

$$w(z)\big|_{L_z} = \varphi(s) + i\psi(s), \quad 0 \leqslant s \leqslant 1 \tag{1.4}$$

on the boundary L_z.

2. CONSTRUCTION OF THE COMPLEX POTENTIAL FOR FLOW AROUND A CIRCLE WITH A SUCTION SLOT

When account is taken of (1.1) and (1.3), condition (1.4) enables us to determine the equation of the boundary of the domain G_w, lying in an infinite sheeted Riemann surface, in the w plane. We shall henceforth consider the single sheet of this surface (Fig. 3a) which corresponds to the domain G_z with a cut along the streamline descending from the trailing edge.

We now introduce the auxiliary plane ζ and, as the canonical domain G_ζ, we will choose the exterior of the unit circle (Fig. 3b). In order to find the dependence of the arc abscissa s of the contour L_z on the arc coordinate γ of the circle $|\zeta| = 1$, it is necessary to construct the complex potential $w = \omega(\zeta)$ for a flow around the unit circle, on the boundary of which the segment MN is an equipotential with a flow rate Q, and the circulation around the circle must be equal to Γ. The domain G_w, with a possible shift, must be the domain of values of this complex potential $\omega(\zeta)$. There will be no shift if one accepts that $\omega(\zeta_a) = 0$, where ζ_a is the flow branch point on the circle. We will denote the free stream velocity in the ζ plane by $u_0e^{i\alpha}$ and represent the function $\omega(\zeta)$ in the form

$$\omega(\zeta) = \omega_1(\zeta) + \omega_2(\zeta) + \omega_3(\zeta); \quad \omega_2(\zeta) = -\frac{Q}{2\pi}\ln q(\zeta), \quad \omega_3(\zeta) = -\frac{\Gamma}{2\pi}\ln g(\zeta)$$

The functions $\omega_1(\zeta)$, $q(\zeta)$, $g(\zeta)$ are regular in the domain G_ζ and have first-order poles at infinity.

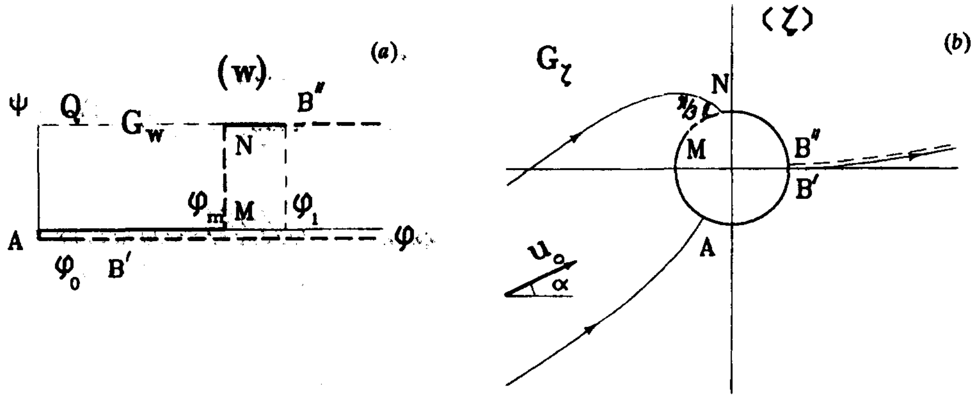

Fig. 3.

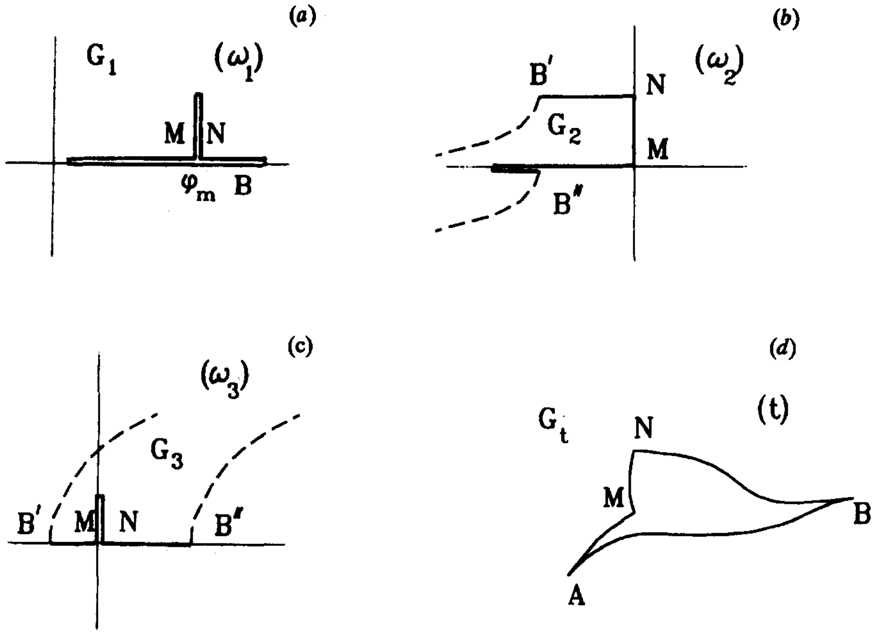

Fig. 4.

The function $\omega_1(\zeta)$, which is the complex potential of circulation-free flow around the unit circle with a zero flow rate through the slot, uniquely maps the exterior of the circle G_ζ into the exterior of the star G_1 (Fig. 4a). The function $\omega_1(\zeta)$ is defined by the formula (see [6], for example)

$$\omega_1(\zeta) = u_0 e^{-i\alpha}\zeta^{-1}(\zeta-\zeta_k)(\zeta-\zeta_n)^{1/2}(\zeta-\zeta_m)^{1/2} + \varphi_m$$

$$\zeta_k = e^{i\gamma_k}, \quad \zeta_n = e^{i\gamma_n}, \quad \zeta_m = e^{i\gamma_m}$$

where u_0, α, γ_k are real constants. The linear dimensions of the star depend on the magnitude of u_0 while the rotation of the star about the centre depends on α and γ_k. However, since this angle is fixed, there is a relation between α and γ_k, namely, $\gamma_k = 2\pi + 2\alpha - (\gamma_n + \gamma_m)/2$.

The function $\omega_2(\zeta)$, which is the complex potential of circulation-free flow with a flow rate Q, maps the exterior of the circle with a cut along the streamline from the point B to infinity into the domain G_2 shown in Fig. 4(b). The function $\omega_3(\zeta)$ is the complex potential of a pure circulatory flow around the unit circle; it maps the exterior of a circle with a cut along the streamline from point B to infinity into the domain G_3 shown in Fig. 4(c). The construction of the functions $\omega_2(\zeta)$ and $\omega_3(\zeta)$ does not present any obvious difficulties. Finally, the complex potential can be written in the form

$$\omega(\zeta)=u_0e^{-i\alpha}\zeta^{-1}(\zeta-\zeta_k)(\zeta-\zeta_n)^{1/2}(\zeta-\zeta_m)^{1/2}+\varphi_m+iQ-$$
$$-\frac{Q}{2\pi}\operatorname{arch}\left(2\frac{1-\zeta_1}{1-\cos\gamma_r}-1\right)-\frac{\Gamma}{2\pi}\operatorname{arch}\left(2\frac{1+\zeta_1}{1+\cos\gamma_r}-1\right) \tag{2.1}$$
$$\zeta_1=(\zeta/\zeta_p+\zeta_p/\zeta)/2,\quad \zeta_p=e^{i\gamma_p},\quad \gamma_p=(\gamma_m+\gamma_n)/2,\quad \gamma_r=(\gamma_m-\gamma_n)/2$$

It should be noted that the coordinates of the ends of the permeable segment, γ_n, γ_m, are unknown on the circle; the modulus u_0 and the direction α of the flow velocity at infinity are also unknown. At the same time, the quantities Q, Γ and φ_1 are known. In order to determine the parameters u_0, α, γ_p, γ_r, γ_a, we use the conditions

$$\omega\left(e^{+0i}\right)=\varphi_1+iQ,\quad \omega(\zeta_a)=0$$
$$\omega'(1)=0,\quad \omega'(\zeta_a)=0,\quad \omega'(\zeta_n)=0 \tag{2.2}$$

which is a system of five non-linear transcendental algebraic equations. The first and second equations fix the values of the complex potential at the meeting and branch points of the flow while the remaining three equations indicate that the velocities (in the ζ plane) are zero at the critical points B, A and N. Note that the fifth and third equations of this system imply the exclusion of patterns for flows around airfoils when the point N to the right along the upper or lower side of the lip. This assertion can be formulated as a condition which is analogous to the Zhukovskii–Chaplygin postulate: of all the theoretically possible flows of an ideal incompressible fluid around an airfoil with a suction slot which is modelled by a cut in an equipotential, smooth flow occurs with a finite velocity at points N and B (Fig. 1b). In fact, this assertion enables us (in a similar manner to the Zhukovskii–Chaplygin postulate) to determine the circulation Γ and the flow rate Q in direct problems.

The formula for the velocity has the form [3]

$$\frac{d\omega}{d\zeta}=u_0e^{i\alpha}\frac{(\zeta-\zeta_a)(\zeta-1)}{\zeta^2}\left(\frac{\zeta-\zeta_n}{\zeta-\zeta_m}\right)^{1/2} \tag{2.3}$$

If we use the notation of (2.3), system (2.2) can be converted to the form

$$\omega(e^{+0i})=\varphi_1+iQ,\quad \omega(\zeta_a)=0,\quad \gamma_a=\pi+2\alpha+\gamma_r$$
$$Q=4\pi u_0\sin(\gamma_r/2)[\sin(\alpha+\gamma_r/2)+\sin(\gamma_p-\alpha)\cos(\gamma_r/2)]$$
$$\Gamma=4\pi u_0\cos(\gamma_r/2)[\sin(\alpha+\gamma_r/2)+\cos(\gamma_p-\alpha)\sin(\gamma_r/2)]$$

The proof of the unique solvability of this system follows from the existence and uniqueness of the function $\omega(\zeta)$.

We will now prove that the function $\omega(\zeta)$ which maps the exterior of a circle with a cut into the domain G_w in the w plane, exists and that it is unique.

We transform the domain G_w (Fig. 3a) with the function $t=f_0^{-1}(w)$, which is the inverse of the function $w=u_0e^{-i\alpha}t+(-Q+i\Gamma)(2\pi)^{-1}\ln t\equiv f_0(t)$, into a certain domain G_t which is the exterior of a closed contour (Fig. 4d). The domain G_t need not be single-sheeted but must be conformally reducible to a single-sheeted domain. According to Riemann's theorem, a conformal mapping of the domain G_t into the domain G_ζ (Fig. 3b) exists with normalization in the neighbourhood of infinity $\zeta=\Phi_0(t)=t+a_0+a_{-1}/t+\ldots$ The required mapping of the domain G_ζ into G_w is then obtained in the form $w=w(\zeta)=f_0(\Phi_0^{-1}(\zeta))$.

We will now prove uniqueness by contradiction. Suppose that two mappings $w=\omega(\zeta)$ and $w=\omega_*(\zeta)$ exist. The function

$$\zeta_*=F(\zeta)=\omega_*^{-1}(\omega(\zeta))=B_1\zeta+B_0+B_{-1}/\zeta+\ldots$$

maps the exterior of one unit circle, conformally and in a one-to-one correspondence, into the exterior of the other unit circle with the preservation of infinity. Such a function has the form $\zeta_*=e^{-i\sigma}\zeta$. However, if the point B is fixed, that is, it is assumed that the point $\zeta=e^{+0i}$ transfers to the point $w=\varphi_1+iQ$, then $\sigma=0$ and $\omega(\zeta)\equiv\omega_*(\zeta)$.

3. SOLUTION OF THE INVERSE BOUNDARY-VALUE PROBLEM

To solve the problem in question it is necessary to establish the relation between the arc abscissae of the required contour and a circle, that is, $s = s(\gamma)$ for $0 \leqslant \gamma \leqslant 2\pi$. This can be done by comparing the complex velocity potentials on the boundaries of the domains G_z and G_ζ: $w(s) = \omega(e^{i\gamma})$, where $w(s)$ is defined by formula (1.4) and $\omega(e^{i\gamma})$ is defined by formula (2.1) when $\zeta = e^{i\gamma}$. Here, unlike the case of an impermeable profile, one must deal with the four segments ($B'A, AM, MN$ and MB'' in Fig. 3a) of the monotonic dependences of φ or ψ on γ and s. The monotonicity and uniqueness of the dependence $s(\gamma)$, $0 \leqslant \gamma \leqslant 2\pi$ follows from this.

We next introduce the function

$$\chi(\zeta) = \ln(dw/dz) - \ln(\zeta - \zeta_a) - (2-\varepsilon)\ln(\zeta - 1) + (3-\varepsilon)\ln\zeta \tag{3.1}$$

which is analytic in the domain G_ζ.

On the boundary $|\zeta| = 1$, the real part of this function is known

$$\operatorname{Re}(\chi(e^{i\gamma})) \equiv S(\gamma) = \ln|v\,(s(\gamma))| - \ln\left(2\left|\sin\frac{\gamma-\gamma_a}{2}\right|\right) - (2-\varepsilon)\ln\left(2\sin\frac{\gamma}{2}\right)$$

Consequently, the function $\chi(\zeta)$ is the solution of the Schwartz problem for the exterior of a circle

$$\chi(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} S(\gamma)\frac{\zeta + e^{i\gamma}}{\zeta - e^{i\gamma}}d\gamma + i\alpha_0$$

It follows that $\alpha_0 = 0$ from the condition $\operatorname{Im}\chi(\infty) = 0$. Using the representation (3.1), after substituting expression (2.3) we find the function

$$z(\zeta) = u_0 e^{-i\alpha}\int_1^{\zeta} e^{-\chi(\zeta)}\left(1-\frac{1}{\zeta}\right)^{\varepsilon-1}\left(\frac{\zeta-\zeta_n}{\zeta-\zeta_m}\right)^{1/2} d\zeta \tag{3.2}$$

which is analytic in the domain G_ζ.

On putting $\zeta = e^{i\gamma}$ in this relation, we obtain a parametric equation of the required contour of the wing. However, the solution which has been constructed does not ensure the closure of the contour L_z and the agreement of the specified magnitude of the velocity V_∞ with that determined during the solution process. (We recall that the boundary of the permeable segment which is modelled by a slot is also included in the contour L_z of the wing.)

The condition for the velocity at infinity and the specified velocity to be identical is written in the same way as in the case of an impermeable contour (see [2], for example)

$$\int_0^{2\pi} S(\gamma)d\gamma = 0 \tag{3.3}$$

The closure condition is

$$\int_0^{2\pi} S(\gamma)e^{i\gamma}d\gamma = \pi\left(1 - \varepsilon + \frac{\zeta_m - \zeta_n}{2}\right) \tag{3.4}$$

If conditions (3.3) and (3.4) are satisfied, the inverse boundary-value problem of the aerodynamics of an airfoil with a suction slot, which is being considered here, has a unique solution. However, if conditions (3.3) and (3.4) turn out to be unsatisfied, the search for the form of the closed contour for an airfoil with suction can be made using the method of quasisolutions [7].

If we put $\gamma_r = 0$, that is, $s_m = s_n$ in the above formulae, we obtain the solution of the inverse boundary-value problem of aerodynamics in the case of an impermeable aerofoil.

The aerodynamic forces acting on an airfoil with a suction slot are calculated using well-known formulae (see [3], for example).

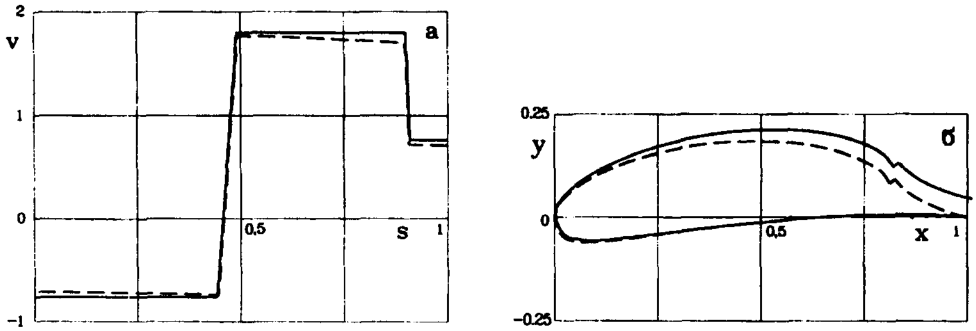

Fig. 5.

$$R_x = \rho Q V_\infty, \quad R_y = \rho \Gamma V_\infty \tag{3.5}$$

where ρ is the fluid density. Since the flow rate $Q > 0$ for an airfoil with suction, R_x is the drag, while, in the case of an airfoil with blowing ($Q < 0$), R_x is the reactive force. Note that, in the case of blowing, formula (3.5) only holds in the case of a potential flow without any breaks in continuity and when the total pressure of the blown jet is identical to the total external pressure.

4. NUMERICAL EXAMPLE

The results of the construction of an airfoil with a lift coefficient greater than two and $\varepsilon = 2$ (an infinitely thin trailing edge) are shown in Fig. 5. The velocity distribution given by the solid line in Fig. 5(a) was taken as the initial velocity distribution and a segment in which the velocity incident on the upper surface ($s_m = 0.90$, $s_n = 0.91$) follows a linear law was chosen for the permeable segment. The velocity on the upper surface up to and after the slot was chosen as being constant. Lighthill [8] has considered airfoils with such velocity distributions when there is no permeable segment. The analogous problem of determining the shape of a diffuser with suction of the flow and with constant velocities on the wall has been solved by Stepanov [9].

In the calculations which are presented here, the length of the permeable segment constituted 1% of the perimeter of the airfoil contour and, of the 200 computational points from which the airfoil contour was constructed, just two points fit on the equipotential. In the example, this boundary is therefore adopted as the section of the normal. We recall that the magnitude of the velocity is dimensionless everywhere, having been divided by V_∞, and the linear dimensions in all calculations and in Fig. 5(a) have been divided by the perimeter of the contour with the exception of the data in Fig. 5(b) where the linear dimensions have been divided by the length of the airfoil chord. Hence, the velocity at infinity $V_\infty = 1$. For the specified velocity distribution $\Gamma = 0.478$ and the flow rate $Q = 0.011$.

An open contour (the solid line in Fig. 5b), around which a flow occurred with $V_\infty = 1.02$, was initially obtained by solving the problem. After using the method of quasisolutions, a closed contour (the dashed line in Fig. 5b) was constructed around which a flow occurred with $V_\infty = 1.0$ at an angle of attack $\beta = 7°$. The profile thickness $t = 21\%$. The lift coefficient for this profile $c_y = 2.09$ and the drag coefficient $c_x = 0.048$. As a result of the quasisolution, the initial velocity distribution (the solid line in Fig. 5a) naturally changed and the modified distribution is shown in Fig. 5(a) by the dashed line.

I wish to thank G. Yu. Stepanov for a number of valuable comments which were taken into account in the final draft of the paper.

This research was carried out with financial support from the Russian Foundation for Basic Research (96-01-00112).

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Translated by E.L.S.